New Generalised Approximation Methods for the Cumulative Distribution Function of arbitrary multivariate Rayleigh Random Variables

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Abstract

Based on our previous works, we revise a simple series expansion for multivariate probability density functions (PDF) of the Rayleigh distribution. From there we derive a similar expression for the cumulative density function (CDF) of multivariate Rayleigh random variables of arbitrary dimension and covariance matrix. The CDF is of particular interest as it can be used to compute outage probabilities of multi-channel wireless systems, for which we provide an example. We compare the performance of the newly proposed approximation to recently proposed methods based on numerical integration methods.

Keywords: Outage Probability, Cumulative Distribution Function, Series Expansion, Multivariate Rayleigh distribution

1 Introduction

The Rayleigh distribution is ubiquitous in the field of engineering, specifically in signal processing. Multivariate Rayleigh random variables play a particularly important role in modelling multi-channel wireless systems. However, as the multivariate extension of the Rayleigh distribution has no known closed form, accurate computations remain troublesome. Approximation methods aimed at certain specific cases (fixed number of channels and correlation structure) of the multivariate case are numerous, and have been proposed for decades [3, 5, 2, 1, 4, 11]. Some recent approximations of the density or other properties extend both dimensionality as well as the range of distribution parameters, but are still subject to considerable limitations [6, 7, 8].

Based on the series expansion introduced by Beard and Tekinay [1] we introduce an extension, wich relaxes parameter assumptions and dimensional restrictions entirely, while preserving computational effort and matching orneven outperforming integration based methods. The computation of the outage probability of multi-channel wireless systems via the cumuluative density function (CDF) is one of the most common applications of the Rayleigh distribution in signal processing. We therefore develop an efficient series expansion that can directly compute these properties while performing similarly or superior to other approaches. Restrictions to neither dimension (e.g. [12]) nor the structure of the covariance matrix (e.g. [16]) will apply in the development of our approximation.

This paper is structured as follows: In Section 2 we perform some preliminary computations that set up the construction of the new PDF expansions, which serve as the foundation of the CDF approximations. We then derive the CDF approximations by integrating over the PDF expansions and create an approximation, thus eliminating the necessity for numerical integration methods. In Section 3 we investigate the computaional performance of the derived approximations against previous approaches that are based on multivariate numerical integration. Section 4 is dedicated to the applications of the newly procured series approximation and a numerical simulation. We close the paper with a brief summary and conclusion in Section 5.

2 Approximation Method

2.1 Preliminary Considerations

We begin with the derivation of the Rayleigh PDF, analogously to previous approaches [15]. The density may be expressed through the distributions of two normal random variables $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ 1.

$$f(x_1, \dots, x_n, y_1, \dots, y_n) = \frac{1}{(2\pi)^n |\Sigma|} \exp\left(-\frac{(x,y)^T \Sigma^{-1} (x,y)}{2} \right) \tag{1}$$

We denote the combined covariance matrix by Σ.

$$\Sigma = \begin{pmatrix}
\sigma_1^2 & 0 & \sigma_1\sigma_2\rho_{1,2} & \dots & \dots & 0 & \sigma_1\sigma_n\rho_{1,n} & 0 \\
0 & \sigma_1^2 & 0 & \sigma_1\sigma_2\rho_{1,2} & & & 0 & \sigma_1\sigma_n\rho_{1,n} \\
\sigma_1\sigma_2\rho_{1,2} & 0 & \sigma_2^2 & \ddots & & & & 0 \\
\vdots & \sigma_1\sigma_2\rho_{1,2} & \ddots & \ddots & & & & \vdots \\
 & & & & & & & \\
\vdots & & & & & & & \\
0 & & & & & \ddots & \ddots & \vdots \\
\sigma_1\sigma_n\rho_{1,n} & 0 & & & & \ddots & \sigma_n & 0 \\
0 & \sigma_1\sigma_n\rho_{1,n} & 0 & \dots & & \dots & 0 & \sigma_n
\end{pmatrix}$$

We may then derive the Rayleigh density function by computing the marginal probability function by integrating:

$$f(r_1, \dots, r_n) = \gamma_{n,K} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{t=1}^{p} \exp\left(a_t \cos(\bar{x}_t)\right) dx_1 \dots dx_n \tag{2}$$

Before moving onto new approximation methods for the Rayleigh PDF and CDF, we recall the ultimate intention. The derivation of the results in this paper all have the same starting point 2. The fundamental difference between the approaches lies in the type of expansion applied to the exponential function of the integrand, to further break down the complex integration into a more computable series expansion. We begin with a rather straight forward Taylor expansion around $z = 0$ which we know as the fundamental characterisation of the exponential function.

2.2 Taylor Approximation

Instead of the Bessel-function expansion proposed by Tekinay and Beard [1] we apply the simpler series definition of the exponential function:

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{3}$$

The integrand needs to be acurately approximated in the range $[0, 2\pi]$. Therefore a simple expansion should be sufficently robust for integration, and will greatly simplify the overall expression and speed up evaluation.

$$
\begin{aligned}
f(r_1,\ldots,r_n) &= \gamma_{n,K}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^{p}\sum_{j_t=0}^{\infty}\frac{(a_t\cos(\bar{x}_t))^{j_t}}{j_t!}dx_1\ldots dx_n\\
&\overset{b_{(t,j_t)}=\frac{a_t^{j_t}}{j_t!}}{=} \gamma_{n,K}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^{p}\sum_{j_t=0}^{\infty}b_{(t,j_t)}\cos(\bar{x}_t)^{j_t}dx_1\ldots dx_n\\
&= \gamma_{n,K}\int_0^{2\pi}\cdots\int_0^{2\pi}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\prod_{t=1}^{p}b_{(t,j_t)}\cos(\bar{x}_t)^{j_t}dx_1\ldots dx_n\\
&= \gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\prod_{t=1}^{p}b_{(t,j_t^*)}\times\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^{p}\cos(\bar{x}_t)^{j_t^*}dx_1\ldots dx_n\\
&= \gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\prod_{t=1}^{p}b_{(t,j_t^*)}\times\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^{p}\left(\frac{\exp(i\bar{x}_t)+\exp(-i\bar{x}_t)}{2}\right)^{j_|t^*}dx_1\ldots dx_n\\
&= \gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\left(\frac{1}{2}\right)^{\sum_{t=1}^{p}j_t^*}\prod_{t=1}^{p}b_{(t,j_t^*)}\\
&\quad\times\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^{p}\sum_{k_t=0}^{j_t^*}\binom{j_t^*}{k_t}\exp(ik_t\bar{x}_t)\exp(-i(j_t^*-k_t)\bar{x}_t)dx_1\ldots dx_n\\
&= \gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\left(\frac{1}{2}\right)^{\sum_{t=1}^{p}j_t^*}\prod_{t=1}^{p}b_{(t,j_t^*)}\times\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^{p}\sum_{k_t=0}^{j_t^*}\binom{j_t^*}{k_t}\exp(i\bar{x}_t(2k_t-j_t^*))dx_1\ldots dx_n\\
&= \gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\left(\frac{1}{2}\right)^{\sum_{t=1}^{p}j_t^*}\prod_{t=1}^{p}b_{(t,j_t^*)}\\
&\quad\times\int_0^{2\pi}\cdots\int_0^{2\pi}\sum_{k_v\in\mathcal{K}}\prod_{t=1}^{p}\left(\binom{j_t^*}{k_t}\exp(i\bar{x}_t(2k_t-j_t^*))\right)dx_1\ldots dx_n\\
&= \gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\left(\frac{1}{2}\right)^{\sum_{t=1}^{p}j_t^*}\prod_{t=1}^{p}b_{(t,j_t^*)}\\
&\quad\times\int_0^{2\pi}\cdots\int_0^{2\pi}\sum_{k_v\in\mathcal{K}}\left(\prod_{t=1}^{p}\binom{j_t^*}{k_t}\right)\exp(i\sum_{t=1}^{p}\bar{x}_t(2k_t-j_t^*))dx_1\ldots dx_n\\
&= \gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\left(\frac{1}{2}\right)^{\sum_{t=1}^{p}j_t^*}\prod_{t=1}^{p}b_{(t,j_t^*)}\\
&\quad\times\sum_{k_v\in\mathcal{K}}\left(\prod_{t=1}^{p}\binom{j_t^*}{k_t}\right)\int_0^{2\pi}\cdots\int_0^{2\pi}\exp(i\sum_{t=1}^{p}\bar{x}_t(2k_t-j_t^*))dx_1\ldots dx_n
\end{aligned}
\tag{4}
$$

Here $k_v=(k_1,\ldots,k_p)$, so that $\mathcal{K}=\{(k_1,\ldots,k_p)|k_1=0,\ldots,j_1^*;\ldots;k_p=0,\ldots,j_p^*\}$, which denotes all possible permutations of index variables. This formulation is once more highly reminiscent of previous results. However, the coefficients $b_{(t,j_t^*)}$ are differently defined, as well as the sum over the integrals. Regardless, we are still able to evaluate the subsequent integral analytically. Dependent on the coefficient in the power, the integral can only result in 0 or 2π. We define $\alpha_t = 2k_t - j_t^*$, and may therefore define the Integral as follows:

$$\text{Int} = (2\pi)^n \underbrace{\prod_{l=1}^{n-1} \left(1 - \prod_{m=1}^{n-1} \mathbb{I}_{\left\{\text{sgn}(a_{l,m})\left(2k_{a_{l,m}} - j^*_{l,m}\right)\right\}}\right)}_{=\text{Int}(j_t^*, k_v)}, \tag{5}$$

where A denotes the matrix of indices $1, \ldots, p$. The matrix A for the n-dimensional Rayleigh distribution is of dimension $n-1$, and thus can be written as follows:

$$A_n = \begin{pmatrix} 1 & n & n+1 & \ldots \\ 2 & -n & 2n-3 & \ldots \\ 3 & -(n+1) & & \ldots \\ \vdots & \vdots & & p \\ n-1 & -2(n-3) & & -p \end{pmatrix}.$$

In this manner the matrix elements are succesively set to integers between 1 and p. We elaborated in our previous work [15] how to determine the non-zero contributions. Since the integral has been analytically resolved, we are therefore left with:

$$f(r_1, \ldots, r_n) = (2\pi)^n \gamma_{n,K} \sum_{j_1=0}^{\infty} \cdots \sum_{j_p=0}^{j_{p-1}} \left(\frac{1}{2}\right)^{\sum_{t=1}^{p} j_t^*} \prod_{t=1}^{p} b_{(t,j_t^*)} \sum_{k_v \in \mathcal{K}} \left(\prod_{t=1}^{p} \binom{j_t^*}{k_t}\right) \text{Int}(j_t^*, k_v). \tag{6}$$

The reverse application of the Cauchy sum reintroduces the original indeces j_t and eliminates the nested sums, providing a more manageable and terse formulation.

2.2.1 Cumulative Distribution Function

Based on the result of the previous section we can now continue to derive the CDF of the Rayleigh distribution by $F(x_1, \ldots, x_n) = \int_0^{x_n} \cdots \int_0^{x_1} f(r_1, \ldots, r_n) dr_1 \ldots dr_n$. We introduce the notation $j^* = (j_1^*, \ldots, j_p^*)$ to summarise all terms not essential for integration.

$$F(x_1,\dots,x_n) = \int_0^{x_n}\cdots\int_0^{x_1}(2\pi)^n\gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\left(\frac{1}{2}\right)^{\sum_{t=1}^p j_t^*}\prod_{t=1}^{p} b_{(t,j_t^*)}\sum_{k_v\in\mathcal{K}}\left(\prod_{t=1}^{p}\binom{j_t^*}{k_t}\right)\mathrm{Int}(j_t^*,k_v)dr_1\dots dr_n$$

$$\overset{\text{Tonelli}}{=}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\underbrace{(2\pi)^n\left(\frac{1}{2}\right)^{\sum_{t=1}^p j_t^*}\sum_{k_v\in\mathcal{K}}\left(\prod_{t=1}^{p}\binom{j_t^*}{k_t}\right)\mathrm{Int}(j_t^*,k_v)}_{=c(j^*)}\int_0^{x_n}\cdots\int_0^{x_1}\gamma_{n,K}\prod_{t=1}^{p}b_{(t,j_t^*)}dr_1\dots dr_n$$

$$=\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}c(j^*)\int_0^{x_n}\cdots\int_0^{x_1}\frac{r_1\cdots r_n}{(2\pi)^n|K|^{1/2}}\prod_{t=1}^{p}\frac{a_t^{j_t^*}}{j_t^*!}\exp\left(-\frac{-1}{|K|}\sum_{i=1}^{n}r_i^2c_{i,i}\right)dr_1\dots dr_n$$

$$\overset{|j-k|=t}{=}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}c(j^*)\int_0^{x_n}\cdots\int_0^{x_1}\frac{r_1\cdots r_n}{(2\pi)^n|K|^{1/2}}\prod_{t=1}^{p}\frac{\left(\frac{-c_t}{|K|}r_lr_k\right)^{j_t^*}}{j_t^*!}\exp\left(-\frac{-1}{|K|}\sum_{i=1}^{n}r_i^2c_{i,i}\right)dr_1\dots dr_n$$

$$=\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}c(j^*)\int_0^{x_n}\cdots\int_0^{x_1}\frac{r_1\cdots r_n}{(2\pi)^n|K|^{1/2}}\prod_{i=1}^{n}r_i^{\bar{j}_i}\prod_{t=1}^{p}\frac{\left(\frac{-c_t}{|K|}\right)^{j_t^*}}{j_t^*!}\exp\left(-\frac{-1}{|K|}\sum_{i=1}^{n}r_i^2c_{i,i}\right)dr_1\dots dr_n$$

$$=\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\frac{c(j^*)}{(2\pi)^n|K|^{1/2}}\prod_{t=1}^{p}\frac{\left(\frac{-c_t}{|K|}\right)^{j_t^*}}{j_t^*!}\int_0^{x_n}\cdots\int_0^{x_1}\prod_{i=1}^{n}r_i^{\bar{j}_i+1}exp\left(-\frac{-1}{|K|}r_i^2c_{i,i}\right)dr_1\dots dr_n$$

$$\overset{ind.}{=}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\frac{c(j^*)}{(2\pi)^n|K|^{1/2}}\prod_{t=1}^{p}\frac{\left(\frac{-c_t}{|K|}\right)^{j_t^*}}{j_t^*!}\prod_{i=1}^{n}\left(\int_0^{x_i}r_i^{\bar{j}_i+1}exp\left(-\frac{-1}{|K|}r_i^2c_{i,i}\right)dx_i\right)$$

$$=\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\frac{c(j^*)}{(2\pi)^n|K|^{1/2}}\prod_{t=1}^{p}\frac{\left(\frac{-c_t}{|K|}\right)^{j_t^*}}{j_t^*!}$$

$$\prod_{i=1}^{n}\left[\frac{2^{\bar{j}_i/2}|K|}{c_{i,i}}\left(\frac{c_{i,i}}{|K|}\right)^{-\bar{j}_i/2}\left(\Gamma\left(\frac{\bar{j}_i}{2}+1\right)-\Gamma\left(\frac{\bar{j}}{2},\frac{c_{i,i}x_i^2}{2|K|}\right)\right)\right]$$

$$=\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\frac{c(j^*)}{(2\pi)^n|K|^{1/2}}\prod_{t=1}^{p}\frac{\left(\frac{-c_t}{|K|}\right)^{j_t^*}}{j_t^*!}\prod_{i=1}^{n}\left(\frac{2^{\bar{j}_i/2}|K|}{c_{i,i}}\left(\frac{c_{i,i}}{|K|}\right)^{-\bar{j}_i/2}\right)$$

$$\prod_{i=1}^{n}\left(\Gamma\left(\frac{\bar{j}_i}{2}+1\right)-\Gamma\left(\frac{\bar{j}_i}{2},\frac{c_{i,i}x_i^2}{2|K|}\right)\right). \tag{7}$$

Here c_t denotes the cofactor matrix value $c_{(i,j)}$ corresponding to the definition of the coefficients a_t. We have denoted $\bar{j}_i$ as the sum of j_t^* values affecting the respective r_i variables. In full this leads to the expression below:

$$
\begin{aligned}
F(x_1, \ldots, x_n) &= \sum_{j_1=0}^{\infty} \cdots \sum_{j_p=0}^{j_{p-1}} \frac{(2\pi)^n \left(\frac{1}{2}\right)^{\sum_{t=1}^{p} j_t^*} \sum_{k_v \in \mathcal{K}} \left(\prod_{t=1}^{p} \binom{j_t^*}{k_t}\right) \mathrm{Int}(j_t^*, k_v)}{(2\pi)^n |K|^{1/2}} \prod_{t=1}^{p} \frac{\left(\frac{-c_t}{|K|}\right)^{j_t^*}}{j_t^*!} \\
&\qquad \prod_{i=1}^{n} \left(2^{\bar{j}_i/2} \left(\frac{c_{i,i}}{|K|}\right)^{-\bar{j}_i/2-1}\right) \prod_{i=1}^{n} \left(\Gamma\left(\frac{\bar{j}_i}{2}+1\right) - \Gamma\left(\frac{\bar{j}}{2}, \frac{c_{i,i}x_i^2}{2|K|}\right)\right) \\
&= \frac{1}{\sqrt{|K|}} \sum_{j_1=0}^{\infty} \cdots \sum_{j_p=0}^{j_{p-1}} \left(\frac{1}{2}\right)^{\sum_{t=1}^{p} j_t^* - \sum_{i=1}^{n} \bar{j}_i/2} \sum_{k_v \in \mathcal{K}} \left(\prod_{t=1}^{p} \binom{j_t^*}{k_t}\right) \mathrm{Int}(j_t^*, k_v) \prod_{t=1}^{p} \frac{\left(\frac{-c_t}{|K|}\right)^{j_t^*}}{j_t^*!} \\
&\qquad \prod_{i=1}^{n} \left(\frac{c_{i,i}}{|K|}\right)^{-\bar{j}_i/2-1} \left(\Gamma\left(\frac{\bar{j}_i}{2}+1\right) - \Gamma\left(\frac{\bar{j}_i}{2}, \frac{c_{i,i}x_i^2}{2|K|}\right)\right)
\end{aligned} \tag{8}
$$

The last line therefore yields a series expansion of the cumulative distribution function for Rayleigh distributions of arbitrary dimension. We like to note that the products largely don't change with different x_i values. The sum over the matrix $\mathcal{K}$ determines which sum terms contribute to the series representation, and remains the same regardless of covariance matrix. The remaining factors are dependent on the values of the covariance matrix, and have to be computed only for non-zero contributions. The gamma functions then need to be evaluated for each new point $(x_1, \ldots, x_n)$. This tiered process separates the setup and evaluation process from each other, and makes the repeated evaluation of the series highly efficient.

2.3 Polynomial Expansion

The series definition of the exponential function constitutes one of the simplest series we can investigate. However, the convergence rate of the Taylor expansion can be matched and surpassed by several other series expansions. Two of which are listed in Equation 9

$$
\exp(z) = \sum_{k=0}^{\infty} \frac{z^{2k-1}(z+2k)}{(2k)!} \qquad \exp(z) = \sum_{k=0}^{\infty} \frac{z^{2k}(z+2k+1)}{(2k+1)!} \tag{9}
$$

These expansions can be introduced into Equation 2 to fashion yet another series expansion for the Rayleigh PDF. The faster convergence rate of these expansion should then translate to the final PDF approximation. Without loss of generality we will use the first version of the polynomial expansion.

$$
\begin{aligned}
f(r_1,\dots,r_n) &= \gamma_{n,K}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^{p}\sum_{j_t=0}^{\infty}\frac{(a_t\cos(\bar{x}_t))^{2j_t-1}((a_t\cos(\bar{x}_t))+2j_t)}{(2j_t)!}dx_1\dots dx_n\\
&= \gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^{p}\frac{(a_t\cos(\bar{x}_t))^{2j_t^*-1}((a_t\cos(\bar{x}_t))+2j_t^*)}{(2j_t^*)!}dx_1\dots dx_n\\
&= \gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\left(\prod_{t=1}^{p}\frac{a_t^{2j_t^*-1}}{(2j_t^*)!}\right)\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^{p}(\cos(\bar{x}_t))^{2j_t^*-1}\prod_{t=1}^{p}((a_t\cos(\bar{x}_t))+2j_t^*)dx_1\dots dx_n\\
&= \gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\left(\prod_{t=1}^{p}\frac{a_t^{2j_t^*-1}}{(2j_t^*)!}\right)\sum_{\xi\in\{0,1\}^p}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^{p}a_t^{\xi_t}\cos(\bar{x}_t)^{2j_t^*-1+\xi_t}(2j_t^*)^{1-\xi_t}dx_1\dots dx_n\\
&= \gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\left(\prod_{t=1}^{p}\frac{a_t^{2j_t^*-1}}{(2j_t^*)!}\right)\sum_{\xi\in\{0,1\}^p}\prod_{t=1}^{p}a_t^{\xi_t}(2j_t^*)^{1-\xi_t}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^{p}\cos(\bar{x}_t)^{2j_t^*-1+\xi_t}dx_1\dots dx_n\\
&= \gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\left(\prod_{t=1}^{p}\frac{a_t^{2j_t^*-1}}{(2j_t^*)!}\right)\sum_{\xi\in\{0,1\}^p}\prod_{t=1}^{p}a_t^{\xi_t}(2j_t^*)^{1-\xi_t}\\
&\quad\times\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^{p}\left(\frac{e^{i\bar{x}_t}+e^{-i\bar{x}_t}}{2}\right)^{2j_t^*-1+\xi_t}dx_1\dots dx_n\\
&= \gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\left(\prod_{t=1}^{p}\frac{a_t^{2j_t^*-1}}{(2j_t^*)!}\right)\sum_{\xi\in\{0,1\}^p}\prod_{t=1}^{p}\frac{a_t^{\xi_t}(2j_t^*)^{1-\xi_t}}{2^{j_t^*-1+\xi_t}}\\
&\quad\times\int_0^{2\pi}\cdots\int_0^{2\pi}\sum_{\rho\in\{-1,1\}^p}\prod_{t=1}^{p}e^{i\bar{x}_t\rho_t(2j_t^*-1+\xi_t)}dx_1\dots dx_n\\
&= \gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\left(\prod_{t=1}^{p}\frac{a_t^{2j_t^*-1}}{(2j_t^*)!}\right)\sum_{\xi\in\{0,1\}^p}\prod_{t=1}^{p}\frac{a_t^{\xi_t}(2j_t^*)^{1-\xi_t}}{2^{j_t^*-1+\xi_t}}\\
&\quad\times\sum_{\rho\in\{-1,1\}^p}\int_0^{2\pi}\cdots\int_0^{2\pi}e^{i\sum_{t=1}^{p}\bar{x}_t\rho_t(2j_t^*-1+\xi_t)}dx_1\dots dx_n.
\end{aligned}
$$

This yields a formulation of the PDF approximation, we can now evaluate. Again, the final step is to determine non-zero sum contributions by the established method, based on the underlying coefficents. Any non-zero integral has the value $(2\pi)^n$. We now compute the corresponding CDF of this series expansion.

$$
f(r_1,\dots,r_n) = (2\pi)^n\gamma_{n,K}\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}}\left(\prod_{t=1}^{p}\frac{a_t^{2j_t^*-1}}{(2j_t^*)!}\right)\sum_{\xi\in\{0,1\}^p}\prod_{t=1}^{p}\frac{a_t^{\xi_t}(2j_t^*)^{1-\xi_t}}{2^{j_t^*-1+\xi_t}}\sum_{\rho\in\{-1,1\}^p}\mathbb{I}(\rho,\xi,j^*) \tag{10}
$$

The indicator $\mathbb{I}\in\{0,1\}$ denotes the outcome of the integration, based on $j^*=(j_1^*,\dots,j_p^*)$,$\rho=(\rho_1,\dots,\rho_p)$ and $\xi=(\xi_1,\dots,\xi_p)$.

2.3.1 Cumulative Distribution Function

As before we integrate the previously acquired PDF approximation, to obtain the CDF.

$$F(x_1,\dots,x_n) = \int_0^{x_1}\cdots\int_0^{x_n} (2\pi)^n \gamma_{n,K} \sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}} \left(\prod_{t=1}^{p} \frac{a_t^{2j_t^*-1}}{(2j_t^*)!}\right) \tag{11}$$

$$\sum_{\xi\in\{0,1\}^p} \prod_{t=1}^{p} \frac{a_t^{\xi_t}(2j_t^*)^{1-\xi_t}}{2^{j_t^*-1+\xi_t}} \sum_{\rho\in\{-1,1\}^p} \mathbb{I}(\rho,\xi,j^*) dr_1\dots dr_n \tag{12}$$

$$= \frac{1}{|K|} \sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}} \left(\prod_{t=1}^{p} \frac{1}{(2j_t^*)!}\right) \sum_{\xi\in\{0,1\}^p} \prod_{t=1}^{p} \frac{(2j_t^*)^{1-\xi_t}}{2^{j_t^*-1+\xi_t}} \sum_{\rho\in\{-1,1\}^p} \mathbb{I}(\rho,\xi,j^*) \tag{13}$$

$$\int_0^{x_1}\cdots\int_0^{x_n} r_1\dots r_n \exp\left(\frac{-1}{|K|}\sum_{i=1}^{n} c_{ii} r_i^2\right) \prod_{t=1}^{p} a_t^{2j_t^*-1\xi_t} dr_1\dots dr_n \tag{14}$$

$$= \frac{1}{|K|} \sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}} \left(\prod_{t=1}^{p} \frac{1}{(2j_t^*)!}\right) \sum_{\xi\in\{0,1\}^p} \prod_{t=1}^{p} \left(\frac{-c_t}{|K|}\right)^{2j_t^*+1-\xi_t} \frac{(2j_t^*)^{1-\xi_t}}{2^{j_t^*-1+\xi_t}} \tag{15}$$

$$\sum_{\rho\in\{-1,1\}^p} \mathbb{I}(\rho,\xi,j^*) \prod_{l}^{n} \int_0^{x_l} \left(r_l \exp\left(\frac{-r_l^2 c_{ll}}{|K|}\right) r_l^{k_l}\right) dx_l. \tag{16}$$

The integration of the innermost sum term is the last remaining non-trivial operation.

$$\begin{aligned}\int_0^{x_l} \left(r_l \exp\left(\frac{-r_l^2 c_{ll}}{|K|}\right) r_l^{k_l}\right) dx_l &= \int_0^{x_l} \left(r_l^{k_l+1} \exp\left(\frac{-r_l^2 c_{ll}}{|K|}\right)\right) dx_l \\ &= \frac{1}{2}\left(\frac{c_{ll}}{|K|}\right)^{-\frac{k_l}{2}-1} \left(\Gamma\left(\frac{k_l}{2}+1\right) - \Gamma\left(\frac{k_l}{2}+1, \frac{c_{ll}x_l^2}{|K|}\right)\right).\end{aligned}$$

Rejoining the integral with the CDF approximation yields the final result below.

$$\begin{aligned}F(x_1,\dots,x_n) = \frac{1}{|K|} \sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}} \left(\prod_{t=1}^{p} \frac{1}{(2j_t^*)!}\right) \sum_{\xi\in\{0,1\}^p} \prod_{t=1}^{p} \left(\frac{-c_t}{|K|}\right)^{2j_t^*+1-\xi_t} \frac{(2j_t^*)^{1-\xi_t}}{2^{j_t^*-1+\xi_t}} \\ \sum_{\rho\in\{-1,1\}^p} \mathbb{I}(\rho,\xi,j^*) \prod_{l}^{n} \frac{1}{2}\left(\frac{c_{ll}}{|K|}\right)^{-\frac{k_l}{2}-1} \left(\Gamma\left(\frac{k_l}{2}+1\right) - \Gamma\left(\frac{k_l}{2}+1, \frac{c_{ll}x_l^2}{|K|}\right)\right).\end{aligned} \tag{17}$$

2.4 Fourier Expansion

We investigate the Bessel function expansion by reintroducing the previously derived PDF function below:

$$f(r_1,\dots,r_n) = \gamma_{n,K} \sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{j_{p-1}} \prod_{t=1}^{p} b_{t,j_t^*} \sum_{\rho\in\{-1,1\}^p} \prod_{t=1}^{p} \mathbb{I}_{\{\alpha_t j_t^* \rho_t = 0\}}. \tag{18}$$

Here $b_{t,j_{t,j?t^*}} = \beta_{t,j_t^*} I_{j_t^*}(|a_t|)$ with $\beta_{t,j_t^*} = 2^{\mathbb{I}_{\{c_t>0\}}}(-1)^{j_t^* \mathbb{I}_{\{j_t^* \neq 0\}}}$. As $a_t = -\frac{c_t}{|K|} r_l r_k$ and $r_l r_k / |K| > 0$, the indicator depends solely on the correlation values c_t. The indicator function inside the sum determines which sum terms are non-zero. As the a_t coefficients contain the variables $r_1, \ldots, r_n$ we need to rearrange terms to find our integration variables for the CDF.

$$
\begin{aligned}
F(x_1, \ldots, x_n) &= \int_0^{x_n} \cdots \int_0^{x_1} \gamma_{n,K} \sum_{j_1=0}^{\infty} \cdots \sum_{j_p=0}^{j_{p-1}} \prod_{t=1}^{p} b_{t,j_t^*} \sum_{\rho \in \{-1,1\}^p} \prod_{t=1}^{p} \mathbb{I}_{\{\alpha_t j_t^* \rho_t = 0\}} dx_1 \ldots dx_n && (19)\\
&= \sum_{j_1=0}^{\infty} \cdots \sum_{j_p=0}^{j_{p-1}} \prod_{t=1}^{p} \beta_{t,j_t^*} \underbrace{\sum_{\rho \in \{-1,1\}^p} \prod_{t=1}^{p} \mathbb{I}_{\{\alpha_t j_t^* \rho_t = 0\}}}_{=R(j_1^*, \ldots, j_p^*)} \int_0^{x_n} \cdots \int_0^{x_1} \gamma_{n,K} \prod_{t=1}^{p} I_{j_t^*}(|a_t|) dx_1 \ldots dx_n && (20)\\
&= \sum_{j_1=0}^{\infty} \cdots \sum_{j_p=0}^{j_{p-1}} \left(\prod_{t=1}^{p} \beta_{t,j_t^*} \right) R(j_1^*, \ldots, j_p^*) \int_0^{x_n} \cdots \int_0^{x_1} \gamma_{n,K} \prod_{t=1}^{p} I_{j_t^*}(|a_t|) dx_1 \ldots dx_n && (21)\\
&= \sum_{j_1=0}^{\infty} \cdots \sum_{j_p=0}^{j_{p-1}} \left(\prod_{t=1}^{p} \beta_{t,j_t^*} \right) R(j_1^*, \ldots, j_p^*) \int_0^{x_n} \cdots \int_0^{x_1} \gamma_{n,K} \prod_{t=1}^{p} \sum_{i_t=0}^{\infty} \frac{\left(\frac{|a_t|}{2} \right)^{2i_t + j_t^*}}{i_t! \Gamma(i_t + j_t^* + 1)} dx_1 \ldots dx_n && (22)\\
&= \sum_{j_1=0}^{\infty} \cdots \sum_{j_p=0}^{j_{p-1}} \left(\prod_{t=1}^{p} \beta_{t,j_t^*} \right) R(j_1^*, \ldots, j_p^*) \int_0^{x_n} \cdots \int_0^{x_1} && (23)\\
&\qquad \gamma_{n,K} \sum_{i_1=0}^{\infty} \cdots \sum_{i_p=0}^{i_{p-1}} \prod_{t=1}^{p} \frac{\left(\frac{|a_t|}{2} \right)^{2i_t^* + j_t^*}}{(i_t^*)! \Gamma(i_t^* + j_t^* + 1)} dx_1 \ldots dx_n && (24)\\
&= \sum_{j_1=0}^{\infty} \cdots \sum_{j_p=0}^{j_{p-1}} \left(\prod_{t=1}^{p} \beta_{t,j_t^*} \right) R(j_1^*, \ldots, j_p^*) \sum_{i_1=0}^{\infty} \cdots \sum_{i_p=0}^{i_{p-1}} \prod_{t=1}^{p} \frac{1}{2^{2i_t^* + j_t^*} (i_t^*)! \Gamma(i_t^* + j_t^* + 1)} && (25)\\
&\qquad \int_0^{x_n} \cdots \int_0^{x_1} \frac{r_1 \ldots r_n}{\sqrt{|K|}} \exp\left(\frac{-1}{|K|} \sum_{i=1}^{n} r_i^2 c_{ii} \right) \prod_{t=1}^{p} \left(\frac{c_t}{|K|} \right)^{2i_t^* + j_t^*} \prod_{l=1}^{n} r^{k_i} dx_1 \ldots dx_n && (26)\\
&= \frac{1}{|K|} \sum_{j_1=0}^{\infty} \cdots \sum_{j_p=0}^{j_{p-1}} \left(\prod_{t=1}^{p} \beta_{t,j_t^*} \right) R(j_1^*, \ldots, j_p^*) \sum_{i_1=0}^{\infty} \cdots \sum_{i_p=0}^{i_{p-1}} \prod_{t=1}^{p} \frac{\left(\frac{c_t}{|K|} \right)^{2i_t^* + j_t^*}}{2^{2i_t^* + j_t^*} (i_t^*)! \Gamma(i_t^* + j_t^* + 1)} && (27)\\
&\qquad \int_0^{x_n} \cdots \int_0^{x_1} \prod_{l=1}^{n} r^{k_l + 1} \exp\left(\frac{-1}{|K|} \sum_{i=1}^{n} r_i^2 c_{ii} \right) dx_1 \ldots dx_n && (28)
\end{aligned}
$$

Now at last we may determine the integrals within the series expansion. Due to indepedence of the integrals we can write the inner integration as follows:

$$
\begin{aligned}
\int_0^{x_n} \cdots \int_0^{x_1} \prod_{l=1}^{n} r^{k_l + 1} \exp\left(\frac{-1}{|K|} \sum_{i=1}^{n} r_i^2 c_{ii} \right) dx_1 \ldots dx_n &= \prod_{l=1}^{n} \int_0^{x_l} r^{k_l + 1} \exp\left(\frac{-c_{ll}}{|K|} r_l^2 \right) dx_l \\
&= \prod_{l=1}^{n} \frac{1}{2} \left(\frac{c_{ll}}{|K|} \right)^{-k_l/2 - 1} \left(\Gamma\left(\frac{k_l + 2}{2} \right) - \Gamma\left(\frac{k_l + 2}{2}, x_l^2 \right) \right).
\end{aligned}
$$

This results in the final formula below:

$$= \frac{1}{|K|} \sum_{j_1=0}^{\infty} \cdots \sum_{j_p=0}^{j_{p-1}} \left(\prod_{t=1}^{p} \beta_{t,j_t^*} \right) R(j_1^*, \ldots, j_p^*) \sum_{i_1=0}^{\infty} \cdots \sum_{i_p=0}^{i_{p-1}} \prod_{t=1}^{p} \frac{\left(\frac{c_t}{|K|}\right)^{2i_t^*+j_t^*}}{2^{2i_t^*+j_t^*}(i_t^*)!\Gamma(i_t^*+j_t^*+1)}$$
$$\prod_{l=1}^{n} \frac{1}{2} \left(\frac{c_{ll}}{|K|} \right)^{-k_l/2-1} \left(\Gamma\left(\frac{k_l+2}{2}\right) - \Gamma\left(\frac{k_l+2}{2}, x_l^2\right) \right).$$

3 Comparison

To test the accuracy of the newly proposed approximation, we compare the approximation with a previously proven accurate version of the PDF series expansion, as well as with a recent integration-based approach. In a previous work we have developed a series expansion based on Bessel functions, which we have shown to be convergent. With a sufficient number of sum terms, we can approximate the true solution to an arbitrary degree, if computation time is secondary.

3.1 Three dimensional case

The example values we intend for the three dimensional example are as follows: $\sigma_1^2 = 0.25, \sigma_2^2 = 0.1, \sigma_3^2 = 0.5$ and the correlation values of $\rho_{1,2} = 0.3, \rho_{1,3} = -0.4$ and $\rho_{2,3} = 0.1$. The covariance matrix can then be written as stated in Equation 29.

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{1,2} & \sigma_1\sigma_2\rho_{1,2} \\ \sigma_1\sigma_2\rho_{1,2} & \sigma_2^2 & \sigma_2\sigma_3\rho_{2,3} \\ \sigma_1\sigma_3\rho_{1,3} & \sigma_2\sigma_3\rho_{2,3} & \sigma_3^2 \end{pmatrix} \tag{29}$$

We have evaluated the different approximations at each point of a three dimensional grid, spanning the hypercube $[0,5]^3$ with a total of 125000 evaluation points. We tested the results against the quasi-true value of the PDF and listed the results in Table 1.

	Name	AAE	REL	AGAE	MAX
Integration	Kronrod	1.9503346E-11	4.6939900E-08	2.4379183E-06	2.4080532E-10
	TOMS614	3.0111506E-08	6.9433501E-05	3.7639383E-03	5.8896215E-07
	Simpson	1.9439856E-11	4.7223663E-08	2.4299820E-06	2.4080532E-10
	Mixed	1.9503344E-11	4.6939877E-08	2.4379180E-06	2.4080532E-10
	Romberg	3.4783528E-11	6.5846685E-07	4.3479410E-06	2.7439149E-10
	Terms				
Bessel Function Expansion	3	1.6256426E-05	5.6008235E-01	2.0320533E+00	2.0966767E-04
	6	1.8165480E-07	1.5522666E-02	2.2706851E-02	4.2003199E-06
	9	1.6144238E-09	2.8107663E-04	2.0180297E-04	6.5966471E-08
	12	1.1423477E-11	3.4827283E-06	1.4279346E-06	8.1955314E-10
	15	6.4546389E-14	3.0641966E-08	8.0682987E-09	7.8266310E-12
	18	2.9238037E-16	1.9732728E-10	3.6547547E-11	5.7371208E-14
	21	1.0674926E-18	9.5416639E-13	1.3343658E-13	3.3293138E-16
	24	3.1332562E-21	3.5393538E-15	3.9165702E-16	1.5619288E-18
	27	7.0473141E-24	1.0044770E-17	8.8091427E-19	1.0842022E-19
	Terms				
Taylor Series Expansion	3	5.4035713E-03	3.0227990E+01	6.7544641E+02	5.1005191E-02
	6	4.3727093E-04	4.9026140E+00	5.4658867E+01	8.1181187E-03
	9	1.9367288E-04	3.2937935E+00	2.4209110E+01	4.3054980E-03
	12	3.5230129E-06	1.6033958E-01	4.4037661E-01	1.7998233E-04
	15	2.8929595E-06	1.6211425E-01	3.6161994E-01	1.7073218E-04
	18	1.2518157E-07	1.6309574E-02	1.5647696E-02	1.0296006E-05
	21	1.9228045E-08	3.1333672E-03	2.4035056E-03	2.7818631E-06
	24	9.1636365E-10	2.5776503E-04	1.1454546E-04	1.9111399E-07
	27	6.0589629E-11	2.2236612E-05	7.5737036E-06	1.7723532E-08

Table 1: Error measures and computation time for different CDF approximation approaches.

We have shown in our previous work how the Bessel function based expansion compares to recent integration approaches. Here we are mainly interested in the comparison of the newly introduced approximation to the other methods. We evaluated up to 27 sum terms in order to match the accuracy of the best integration approaches. Naturally, the series expansion is not limited, and can be increased to any desired accuracy. While the convergence rate of the expansion is considerably slower than that of the Bessel function expansion, the true advantage lies in its simplicity. Rather than having to evaluate numerous Bessel functions (which has to be done via series expansion or by numerical means itself), we can simply evaluate elementary functions, resulting in advantageous evaluation times. However, since we already have a well-performing PDF approximation, we are focused on the applicability of the CDF series expansion. The Taylor expansion benefits from its simplicity. The integral possesses an analytical solution, thus the CDF can be evaluated almost in the same manner as the PDF, saving the trouble of multivariate numerical integration. The result of the same test are listed in Table **??**.

	TYPE	**AAE**	**REL**	**AGAE**	**MAX**	**TIME (s)**
Integration	Kronrod	2.21487617E-10	8.32495896E-08	2.76859522E-08	5.64524982E-10	465.18500
	TOMS614	3.50063212E-07	1.16583701E-04	4.37579015E-05	8.65269964E-07	734.11500
	Simpson	2.20643529E-10	8.30729220E-08	2.75804411E-08	5.63306179E-10	3174.95500
	Mixed	2.21487598E-10	8.32495861E-08	2.76859497E-08	5.64525093E-10	93.59000
	Romberg	1.61332637E-10	7.22673027E-08	2.01665796E-08	5.86725224E-10	921.67100
	Terms					
Taylor Series Expansion	3	4.52660723E-02	6.20384842E+00	5.65825904E+00	1.96297952E-01	0.04100
	6	4.57357190E-03	5.51072582E-01	5.71696488E-01	3.06116569E-02	0.11000
	9	2.30087109E-03	2.60778143E-01	2.87608887E-01	1.72746333E-02	0.24200
	12	1.40297236E-04	1.58233657E-02	1.75371545E-02	1.75059171E-03	0.50600
	15	1.20679456E-04	1.27974222E-02	1.50849320E-02	1.49917424E-03	0.85600
	18	3.06400846E-06	3.41917754E-04	3.83001058E-04	4.20631906E-05	1.41800
	21	6.21738702E-06	6.39626047E-04	7.77173377E-04	1.18353178E-04	2.08900
	24	3.48636411E-07	3.55184783E-05	4.35795514E-05	6.65725151E-06	3.05300
	27	2.95914961E-07	2.99921992E-05	3.69893702E-05	7.79250505E-06	4.26600
	30	3.14789791E-08	3.16392983E-06	3.93487239E-06	9.43384173E-07	5.85700
	33	1.18458305E-08	1.19304465E-06	1.48072881E-06	4.00846585E-07	7.71700
	36	1.58621263E-09	1.60060516E-07	1.98276578E-07	6.05939035E-08	9.93200
	39	3.79293404E-10	3.91619710E-08	4.74116755E-08	1.53232012E-08	12.55600
	42	5.37208262E-11	6.49984091E-09	6.71510327E-09	2.25540042E-09	15.51400
	45	1.83614732E-11	2.96563489E-09	2.29518415E-09	4.71159667E-10	18.81900
	48	8.69508640E-12	1.99775864E-09	1.08688580E-09	8.34170510E-11	22.81800
	51	9.92902113E-12	2.12125057E-09	1.24112764E-09	8.41227088E-11	27.18811

Table 2: Error measures and evaluation time

In this comparison we can see the potential of the CDF representation. As any previous iteration of a CDF approximation is reliant on integration to some degree, all these approaches are limited by the numerical capabaility of the integration method used. The series expansion has the advantage of not requiring integration, and can compute the CDF much faster than all tested approaches. Due to the slower convergence speed we need 40+ iterations to surpass the integration based accuracy. However, while matching the accuracy, still only a fraction of the computational effort is necessary compared to the multivariate numerical integrals.

4 Applications

The most immediate application, for which we developed the series expansions of the previous chapter, is the outage probability of an n-dimensional wireless system (e.g. n channel systems). For this we refer to the definition of the outage probility of such a system as introduced by Chen and Tellambura in [5]:

$$\begin{aligned} P_{out}(\gamma_{th}) &= \int_0^{\sqrt{\frac{\gamma_{th}\Sigma_{(1,1)}}{\gamma_1}}} \int_0^{\sqrt{\frac{\gamma_{th}\Sigma_{(2,2)}}{\gamma_2}}} \int_0^{\sqrt{\frac{\gamma_{th}\Sigma_{(3,3)}}{\gamma_3}}} dr_1 dr_2 dr_3 \\ &= F_R\left(\sqrt{\frac{\gamma_{th}\Sigma_{(1,1)}}{\gamma_1}}, \sqrt{\frac{\gamma_{th}\Sigma_{(2,2)}}{\gamma_2}}, \sqrt{\frac{\gamma_{th}\Sigma_{(3,3)}}{\gamma_3}}\right) \end{aligned} \tag{30}$$

Note that we use the inverted value γ/γ_{th} as x-axis value. This means we may evaluate the outage probabaility by the series expansion of the previous section. We once again use the covariance matrix described in Equation 29. We compute outage probabilities for 1000 steps of the threshold value γ_{th} ranging from 0 to 200. We have added the result of an empirical CDF retrieved by repeatedly sampling the norm $\sqrt{X^2+Y^2}$ of two random variables X and Y with multivariate normal distribution $\mathcal{N}(0,\Sigma)$

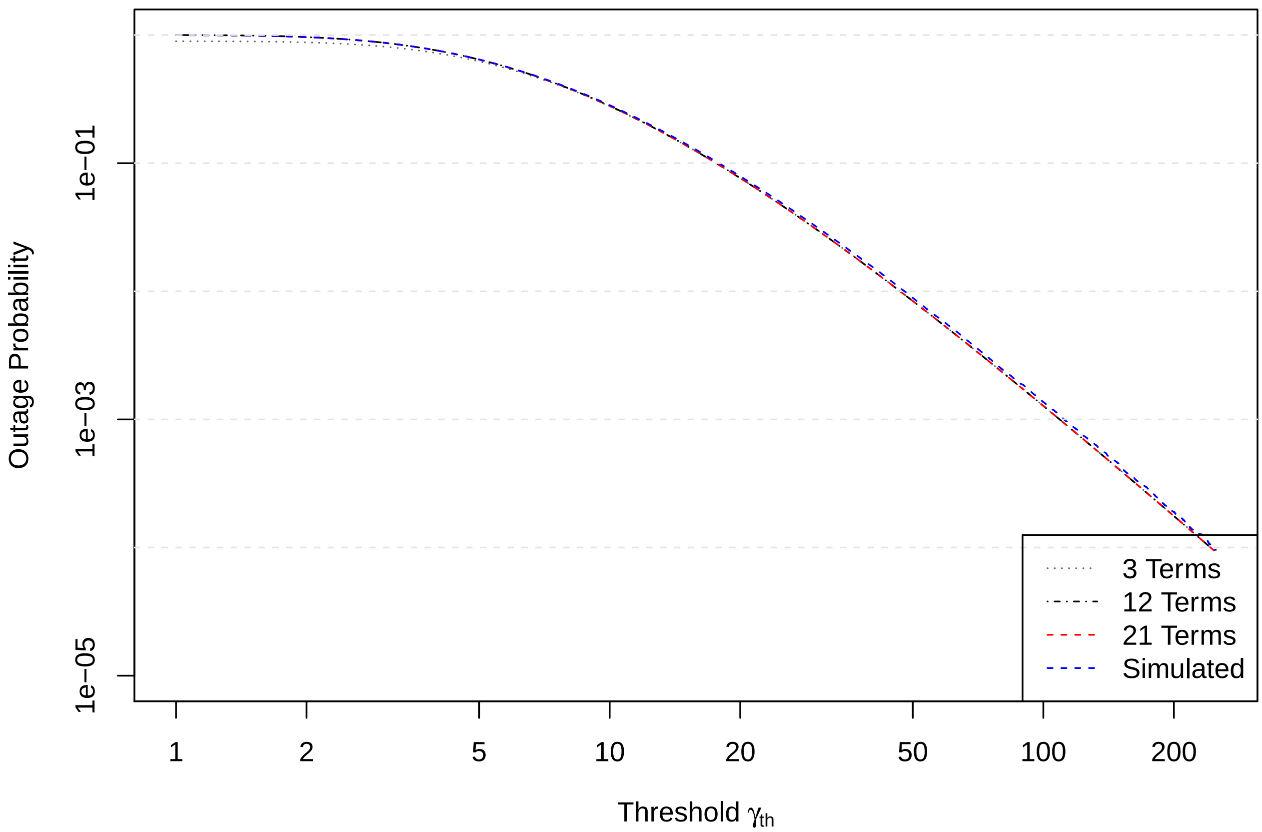

Figure 1: Progression of additional series terms in the outage probability vs threshold values.

In Figure 1 we see the influence of additional series terms on the outage probability as described in Equation 30. The approximation converges rather quickly, as the numerical experiments have proven, and the evaluation is simple and fast. There is little to no visible difference between the simulated and computed results.

This shows the immediate and intended applicability of the CDF series expansion in signal processing. While there may be further applications that can make direct use of either the CDF approximations or the PDF approximations of previous works [15], we have limited ourselves to the most immediate application, and will address further applications in future work.

5 Conclusion

In this paper we have derived several new series representations of the multivariate Rayleigh PDF and CDF by extending previous series approximations. In computational simulations we tested the accuracy of the new series against existing approximations, that still involve numerical integration to varying degrees. We found that avoiding the repeated numerical evaluation of these integrals greatly increased the evaluation speed and therefore practical value. A series expansion for the multivariate Rayleigh CDF has (to the best of our knowledge) not been proposed thus far. The generalised method of approximating multivariate Rayleigh CDFs of arbitrary dimension and correlations lends tremendous flexibility and range to potential applications. As the CDFs are for example commonly needed for outage probabilities in wireless communications systems, the approximations may lead to exciting new applications. For future work the properties of the proposed approach in other applications and settings may be of interest, such as the behaviour under outdated CSI conditions on the performance of the approximation [**?**, **?**].

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